Convexity of the images of small balls through perturbed convex multifunctions

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Abstract

In the present paper, the following convexity principle is proved: any closed convex multifunction, which is metrically regular in a certain uniform sense near a given point, carries small balls centered at that point to convex sets, even if it is perturbed by adding $\mathrm{C}^{1,1}$ smooth mappings with controlled Lipschizian behaviour. This result, which is valid for mappings defined on a subclass of uniformly convex Banach spaces, can be regarded as a set-valued generalization of the Polyak convexity principle. The latter, indeed, can be derived as a special case of the former. Such an extension of that principle enables one to build large classes of nonconvex multifunctions preserving the convexity of small balls. Some applications of this phenomenon to the theory of set-valued optimization are proposed and discussed.

Keywords: convex multifunction, uniformly convex Banach space, modulus of convexity, metric regularity, Polyak convexity principle, set-valued optimization.

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1 Introduction

Treating problems from mathematical programming, optimal control and from several areas of mathematical economics yields a tremendous demand of convexity. Convexity assumptions on problem data often strenghten the analysis tools and trigger the application of special approaches, otherwise not practicable. Even though such a demand has led to deepen our knowledge about convexity and then to develop expanding branches of convex analysis, many fundamental issues about convexity still remain to be investigated. In the author's opinion, one of such issues concerns the behaviour of convex sets under nonlinear transformations. Indeed, not much seems to be known so far about those sets whose image through nonlinear mappings is convex. The existing results on this question can be schematically classified as "around a point" (local) results or as "on set" (nonlocal) results. As an example of nonlocal result the Lyapunov convexity theorem on the range of a vector measure occupies a prominent place (see [14]). It found notable applications in control theory and mathematical economics (see [1, 17]). Other examples of global results are, for instance, those in [5, 21, 22]. As an example of local result, the Polyak convexity principle is certainly to be mentioned (see [18, 19]). Like the Lyapunov's theorem, it revealed to be useful in several topics of optimization and control theory, by providing conditions upon which nonlinear mappings carry small balls around a point to convex sets.

The present paper aims at bringing some contributions in the same vein as the Polyak convexity principle, but entering now the realm of set-valued mappings. The starting point of the analysis here proposed is the well-known fact that convex multifunctions (i.e. set-valued mappings with convex graph) carry any convex set to a convex set. If considering the category, whose objects are convex sets, this class of mappings seem to naturally play the role of category morphisms. Unfortunately, by simple examples it is readily realized that, when adding a nonlinear single-valued mapping to a convex multifunction, in general the convex graph property of the latter is broken. Thus the question arises under which conditions mappings, obtained by perturbing convex multifunctions by nonlinear mappings, still carry small balls to convex sets. The main result of this paper provides an answer to this problem. It states that, if to a convex multifunction, which is metrically regular near a reference point uniformly over its image, a $\mathrm{C}^{1,1}$ mapping is added, whose Lipschitzian behaviour is controlled by the modulus of regularity of the former, then the resulting set-valued mapping preserves the convexity of small balls around the reference point of its domain. In fact, this result can be regarded as an extension of the Polyak convexity principle to a large class of set-valued mappings. As it happens for its single-valued counterpart, it is valid for mappings defined on uniformly convex Banach spaces having second order polynomial modulus of convexity. This class of spaces includes, for instance, all Hilbert spaces. The proof combines a nice property, coming from the rotund geometry of balls in the aforementioned class of Banach spaces, with a convex solvability behaviour of set-valued mappings, that are perturbed as described. The latter is a consequence of the persistence of metric regularity under additive Lipschitz perturbations, a well-known phenomenon in variational analysis, which has revealed to be useful in various contexts related to the solution stability and sensitivity for generalized equations (see [9, 16]).

The contents of the paper are organized as follows. In Section 2 some tools, mainly from geometric functional analysis and from nonlinear analysis, that are needed for establishing the main result are recalled. In particular, in Subsection 2.3 a strenghtened notion of metric regularity for set-valued mappings is introduced. Several classes of multifunctions satisfying such a special property are exhibited, while it is observed that the original notion of metric regularity is weaker (in the sense that it holds more generally). In Section 3 the main result is proved and commented. Then, it is shown how from this wider convexity principle the

Polyak's one can be derived, as a special case. Section 4 is reserved to illustrate an application of the main result to a topic from set-valued optimization. More precisely, a class of optimization problems is considered, whose set-valued objective is expressed as a sum of a single-valued and a set-valued mapping. This structure in the objective mapping may model noise effects on vector optimization problems. In this context, the convexity principle, under certain additional assumptions, leads first of all to establish the existence of efficient pairs for localizations of an unconstrained problem, and then to achieve optimality conditions based on the Lagrangian scalarization.

2 Tools from nonlinear analysis

2.1 Uniformly convex Banach spaces

The analysis of the posed problem will be carried out in the particular setting of the uniformly convex real Banach spaces. This because the main result presented in the paper essentially rely on certain geometrical features of this specific class of Banach spaces, features that are related to the rotundity of the balls. The rotundity property of a ball in a Banach space $(\mathbb{X}, \|\cdot\|)$ can be quantitatively described by means of the function $\delta_{\mathbb{X}} : [0, 2] \longrightarrow [0, 1]$, defined by

$$\delta_{\mathbb{X}}(\epsilon) = \inf \left\{ 1 - \left\| \frac{x_1 + x_2}{2} \right\| : \ x_1,\, x_2 \in \mathbb{B},\ \|x_1 - x_2\| \geq \epsilon \right\},$$

which is called the *modulus of convexity* of $(\mathbb{X}, \|\cdot\|)$ [1]. $\mathbb{B}$ stands for the closed unit ball, centered at the null vector $\mathbf{0}$ of $\mathbb{X}$. Notice that $\delta_{\mathbb{X}}$ is not invariant under equivalent renormings of $\mathbb{X}$. Such a notion allows one to define the class of uniformly convex Banach spaces, whose introduction is due to J.A. Clarkson (see, for instance, [7, 10, 15]).

Definition 2.1 A Banach space $(\mathbb{X}, \|\cdot\|)$ is called *uniformly convex* (or, *uniformly rotund*) if it is $\delta_{\mathbb{X}}(\epsilon) > 0$ for every $\epsilon \in (0, 2]$.

In what follows, the modulus of convexity of a (uniformly convex) Banach space is said to be of the *(polynomial) second order* if there exists $c > 0$ such that

$$\delta_{\mathbb{X}}(\epsilon) \geq c\epsilon^2, \quad \forall \epsilon \in [0, 2].$$

The class of uniformly convex real Banach spaces with second order modulus of convexity reveals to be the proper setting, in which to develp the analysis of the problem at the issue. Throughout the paper, this class will be indicated by UC_2.

Example 2.1 (e_1) By means of elementary considerations, the modulus of convexity of a Hilbert space $\mathbb{H}$ can be calculated to amount to

$$\delta_{\mathbb{H}}(\epsilon) = 1 - \sqrt{1 - \frac{\epsilon^2}{4}}, \quad \forall \epsilon \in [0, 2].$$

Therefore, every Hilbert space is uniformly convex, with a second order modulus of convexity, such that $0 < c \leq 1/8$, i.e. belongs to the class UC_2.

[1] Equivalent definitions of the modulus of convexity can be found in [10].

(e_2) More generally, such Banach spaces as l^p, L^p, and W^p_m, with $1 < p < 2$, are known to have a modulus of convexity satisfying the relation

$$\delta_{l^p}(\epsilon) = \delta_{L^p}(\epsilon) = \delta_{W^p_m}(\epsilon) > \frac{p-1}{8}\epsilon^2, \quad \forall \epsilon \in (0, 2].$$

Therefore, they also are examples of spaces of class UC_2 (see, for instance, [10]).

Remark 2.1 (r_1) Concerning the notion of uniform convexity, a caveat is due: even finite-dimensional Banach spaces may fail to be uniformly convex. Consider, for instance, $\mathbb{R}^2$ equipped with the Banach space structure given by the norm $\|\cdot\|_\infty$.

(r_2) It was proved that the modulus of convexity $\delta_{\mathbb{X}}$ of any real Banach space, having dimension greater than 1, admits the following estimate from above

$$\delta_{\mathbb{X}}(\epsilon) \leq 1 - \sqrt{1 - \frac{\epsilon^2}{4}}, \quad \forall \epsilon \in [0, 2].$$

This implies that the second order polynomial is a maximal one.

(r_3) Recall that, according to the Milman-Pettis theorem, every uniformly convex Banach space is reflexive, but the converse is false (see, for instance, [10]).

For further material about uniformly convex Banach spaces, see [10, 15]. In the following lemma, whose proof can be found in [25] (Lemma 2.4), a key property of balls in any uniformly convex Banach space of class UC_2 is stated, in view of a subsequent application. Throughout the paper, given an element $x \in \mathbb{X}$ and a real $r \geq 0$, $\mathrm{B}(x, r)$ denotes the closed ball centered at the point x, with radius r.

Lemma 2.1 *Let $(\mathbb{X}, \|\cdot\|)$ belong to UC_2, with modulus of convexity $\delta_{\mathbb{X}}(\epsilon) \geq c\epsilon^2$, for some $c > 0$. Then, for every x_0, x_1, $x_2 \in \mathbb{X}$ and $r > 0$, with x_1, $x_2 \in \mathrm{B}(x_0, r)$, it holds*

$$\mathrm{B}\left(\frac{x_1 + x_2}{2}, \frac{c\|x_1 - x_2\|^2}{r}\right) \subseteq \mathrm{B}(x_0, r).$$

2.2 Smooth mappings and Lipschitzian properties

Let $f : \Omega \longrightarrow \mathbb{Y}$ be a mapping between real Banach spaces, where Ω is a nonempty open subset of $\mathbb{X}$. Its Gâteaux derivative at $\bar{x} \in \Omega$ is denoted by $\mathrm{D}f(\bar{x})$. Let us indicate by $(\mathcal{L}(\mathbb{X}, \mathbb{Y}), \|\cdot\|_{\mathcal{L}})$ the Banach space of all linear bounded operators between $\mathbb{X}$ and $\mathbb{Y}$, equipped with the operator norm. If f admits Gâteaux derivative at each point of Ω and the mapping $\mathrm{D}f : \Omega \longrightarrow \mathcal{L}(\mathbb{X}, \mathbb{Y})$, defined by $x \mapsto \mathrm{D}f(x)$, is norm-to-$\|\cdot\|_{\mathcal{L}}$ continuous, then f is said to be of class $\mathrm{C}^1(\Omega)$. Remember that if $f \in \mathrm{C}^1(\Omega)$, f is in particular strictly differentiable at each point of Ω. If, furthermore, the mapping $\mathrm{D}f$ is Lipschitz continuous on Ω, f is said to be of class $\mathrm{C}^{1,1}(\Omega)$. In such a case, the infimum of all constants $\kappa > 0$ such that

$$\|\mathrm{D}f(x_1) - \mathrm{D}f(x_2)\|_{\mathcal{L}} \leq \kappa\|x_1 - x_2\|, \quad \forall x_1, x_2 \in \Omega,$$

will be indicated by $\mathrm{Lip}(\mathrm{D}f, \Omega)$. In the same setting, given a point $\bar{x} \in \Omega$, let us define the value

$$\mathrm{lip}\, f(\bar{x}) = \limsup_{\substack{u, x \to \bar{x} \\ u \neq x}} \frac{\|f(u) - f(x)\|}{\|u - x\|}$$

the Lipschitz modulus of f at $\bar{x}$. Clearly, $\operatorname{lip} f(\bar{x}) < \infty$ iff f is locally Lipschitz in a neighbourhood of $\bar{x}$. In particular, if $f \in \mathrm{C}^1(\operatorname{int} \mathrm{B}(\bar{x}, r))$ for some $r > 0$, then one has $\operatorname{lip} f(\bar{x}) = \|\mathrm{D}f(\bar{x})\|_{\mathcal{L}} < \infty$. Throughout the paper, the convention is adopted that, whenever $\|\mathrm{D}f(\bar{x})\|_{\mathcal{L}} = 0$ or $\operatorname{lip} f(\bar{x}) = 0$, the symbols $\|\mathrm{D}f(\bar{x})\|_{\mathcal{L}}^{-1}$ and $\operatorname{lip} f(\bar{x})^{-1}$ stand for $+\infty$.

This short subsection is concluded by a lemma, stating an estimate for $\mathrm{C}^{1,1}$ smooth mappings that will be crucially employed in the proof of the main result. For its proof, the reader is referred to [25] (Lemma 2.7).

Lemma 2.2 *Let $f : \mathbb{X} \longrightarrow \mathbb{Y}$ be a mapping between Banach spaces, let $U \subseteq \mathbb{X}$, let $\Omega \subseteq \mathbb{X}$ be an open set such that $\Omega \supseteq U$, and let $x_1, x_2 \in U$, with $[x_1, x_2] \subseteq U$. If $f \in \mathrm{C}^{1,1}(\Omega)$, then it holds*

$$\left\| \frac{f(x_1) + f(x_2)}{2} - f\left(\frac{x_1 + x_2}{2}\right) \right\| \leq \frac{\operatorname{Lip}(\mathrm{D}f, U)}{8} \|x_1 - x_2\|^2.$$

2.3 Convex multifunctions and their metric regularities

Throughout the paper, given a subset A of a Banach space and a point x in the same space, $\operatorname{dist}(x, A) = \inf_{a \in A} \|a - x\|$ denotes the distance of x from A. The notion of metric regularity, along with its equivalent reformulations, is recognized as an important tool in the variational analysis of set-valued mappings. Recall that, given a set-valued mapping $G : \mathbb{X} \rightrightarrows \mathbb{Y}$ between real Banach spaces, G is said to be metrically regular at $\bar{x}$, for $\bar{y}$, with $(\bar{x}, \bar{y}) \in \operatorname{gph} G = \{(x, y) \in \mathbb{X} \times \mathbb{Y} : \ y \in G(x)\}$, provided that there exist positive constants κ, δ, and ζ such that

$$\operatorname{dist}\left(x, G^{-1}(y)\right) \leq \kappa \operatorname{dist}(y, G(x)), \quad \forall x \in \mathrm{B}(\bar{x}, \delta), \forall y \in \mathrm{B}(\bar{y}, \zeta). \tag{1}$$

The constant

$$\operatorname{reg} G(\bar{x}|\bar{y}) = \inf\{\kappa \in (0, +\infty) : \ (1) \text{ holds for some } \delta \text{ and } \zeta\}$$

is usually called regularity modulus of G at $\bar{x}$, for $\bar{y}$. Several aspects of the theory of metric regularity are exposed in recent monographs (among the others, see [6, 9, 13, 16, 20]).

In what follows, a metric regularity property, which is stronger than the original metric regularity at a reference pair, will be needed. Below, given a real $r \geq 0$ and a subset $A \subset \mathbb{Y}$, by $\mathrm{B}(A, r) = \{y \in \mathbb{Y} : \operatorname{dist}(y, A) \leq r\}$ the r-enlargement of A will be indicated.

Definition 2.2 A set-valued mapping $G : \mathbb{X} \rightrightarrows \mathbb{Y}$ is said to be *metrically regular at* $\bar{x} \in \operatorname{dom} G = \{x \in \mathbb{X} : G(x) \neq \varnothing\}$, *for* $G(\bar{x})$, if there exist positive constants κ, δ, and ζ such that

$$\operatorname{dist}\left(x, G^{-1}(v)\right) \leq \kappa \operatorname{dist}(v, G(x)), \quad \forall x \in \mathrm{B}(\bar{x}, \delta), \forall v \in \mathrm{B}(G(\bar{x}), \zeta). \tag{2}$$

The constant

$$\operatorname{reg} G(\bar{x}) = \sup_{y \in G(\bar{x})} \operatorname{reg} G(\bar{x}|y)$$

will be used as a *regularity modulus* of G at $\bar{x}$, for $G(\bar{x})$.

From (2) one immediately sees that metric regularity at $\bar{x}$, for $G(\bar{x})$, implies (and is actually equivalent to) the metric regularity of G at $\bar{x}$, for every $y \in G(\bar{x})$, with the same constants κ, δ, and ζ in (1). To the contrary, metric regularity at each pair $\bar{x}$ and $y \in G(\bar{x})$, without uniformity on the values of κ, δ, and ζ fails in general to imply metric regularity of G at $\bar{x}$, for $G(\bar{x})$. The example below illustrates such an occurence.

Example 2.2 Consider the function $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$, defined by

$$g(y_1, y_2) = y_1 y_2,$$

and, as a multifunction $G : \mathbb{R} \rightrightarrows \mathbb{R}^2$, its inverse mapping

$$G(x) = g^{-1}(x) = \{y = (y_1, y_2) \in \mathbb{R}^2 : \; y_1 y_2 = x\}.$$

Set $\bar{x} = 0$ and $\bar{y} = (\bar{y}_1, \bar{y}_2) = (0, 0)$. Clearly, $G(0) = \{y \in \mathbb{R}^2 : \; y_1 y_2 = 0\}$ is represented in the Euclidean plane as the union of the two coordinate axes. Observe that, since $g \in \mathrm{C}^1(\mathbb{R}^2)$, then g is locally Lipschitz near each point $y \in G(0)$. By consequence, according to Theorem 1.49 in [16], its inverse mapping G turns out to be metrically regular at 0, for each $y \in G(0)$. Now, let κ, δ and ζ be arbitrary, but fixed, positive reals. One has

$$\mathrm{B}\,(G(0), \zeta) = [\mathbb{R} \times (-\zeta, \zeta)] \cup [(-\zeta, \zeta) \times \mathbb{R}].$$

Notice that, if $x \in (-\delta, \delta)$ is close enough to 0, it is

$$G(x) \subseteq \mathrm{B}\,(G(0), \zeta)\,.$$

Let $x_\delta > 0$ be such a point. Then, if $v = (v_1, v_2) \in \mathrm{B}\,(G(0), \zeta) \cap \mathbb{R}^2_+$, with $\mathbb{R}^2_+$ denoting the nonnegative cone in $\mathbb{R}^2$, one sees that

$$\mathrm{dist}\,(v, G(x_\delta)) < \zeta.$$

Thus, choose $\bar{v}_2 = \zeta/2$ and $\bar{v}_1$ in such a way that $\bar{v}_1 \bar{v}_2 - x_\delta > \kappa\zeta$, i.e.

$$\bar{v}_1 > \frac{2(\kappa\zeta + x_\delta)}{\zeta}.$$

It remains true that $\bar{v} \in \mathrm{B}\,(G(0), \zeta)$, but one finds

$$\mathrm{dist}\left(x_\delta, G^{-1}(\bar{v})\right) = |x_\delta - \bar{v}_1 \bar{v}_2| > \kappa\zeta > \kappa \mathrm{dist}\,(v, G(x_\delta))\,.$$

So, inequality (2) is clearly violated.

Nonetheless, under additional assumptions on G, the metric regularity at $\bar{x}$, for each point of $G(\bar{x})$, can imply the metric regularity at $\bar{x}$, for $G(\bar{x})$. This happens, for instance, with multifunctions taking compact values, as established in the next proposition. Throughout the paper, given a subset A of a Banach space, by $\mathrm{int}\, A$ the (topological) interior of A is denoted.

Proposition 2.1 *Let $G : \mathbb{X} \rightrightarrows \mathbb{Y}$ be a set-valued mapping between Banach spaces, and let $\bar{x} \in \mathrm{dom}\, G$. If G is metrically regular at $\bar{x}$, for every $y \in G(\bar{x})$, and $G(\bar{x})$ is compact, then G is metrically regular at $\bar{x}$, for $G(\bar{x})$.*

Proof. By virtue of the metric regularity of G at $\bar{x}$, for every $y \in G(\bar{x})$, there exist positive δ_y, ζ_y and κ_y such that

$$\mathrm{dist}\left(x, G^{-1}(v)\right) \le \kappa_y \,\mathrm{dist}\,(v, G(x))\,, \quad \forall x \in \mathrm{B}\,(\bar{x}, \delta_y)\,,\; \forall v \in \mathrm{B}\,(y, \zeta_y)\,. \tag{3}$$

Notice that the family $\{\mathrm{int}\,\mathrm{B}\,(y, \zeta_y/2) : \; y \in G(\bar{x})\}$ forms an open covering of $G(\bar{x})$. Since $G(\bar{x})$ has been supposed to be compact, this family must admit a finite subfamily still covering $G(\bar{x})$, say $\{\mathrm{int}\,\mathrm{B}\,(y_i, \zeta_{y_i}/2) : y_i \in G(\bar{x}),\; i = 1, \dots, m\}$. Thus, it is possible to define the following positive constants

$$\delta = \min\{\delta_{y_i} : \; i = 1, \dots, m\}, \qquad \zeta = \min\{\zeta_{y_i} : \; i = 1, \dots, m\}, \qquad \text{and} \qquad \kappa = \max\{\kappa_{y_i} : \; i = 1, \dots, m\}.$$

Now, if $v \in \mathrm{B}\left(G(\bar{x}), \zeta/3\right)$, there must exist $y \in G(\bar{x})$ such that $d(v, y) < \zeta/2$. Since it is

$$y \in G(\bar{x}) \subseteq \bigcup_{i=1}^{m} \operatorname{int} \mathrm{B}\left(y_i, \zeta_{y_i}/2\right),$$

then for some index $i^* \in \{1, \ldots, m\}$ one has $y \in \operatorname{int} \mathrm{B}\left(y_{i^*}, \zeta_{y_{i^*}}/2\right)$. It follows

$$d(v, y_{i^*}) \leq d(v, y) + d(y, y_{i^*}) < \frac{\zeta}{2} + \frac{\zeta_{y_{i^*}}}{2} \leq \zeta_{y_{i^*}}.$$

Hence it is possible to invoke inequality (3), in the case $y = y_{i^*}$. Consequently, one obtains

$$\operatorname{dist}\left(x, G^{-1}(v)\right) \leq \kappa \operatorname{dist}\left(v, G(x)\right), \quad \forall x \in \mathrm{B}\left(\bar{x}, \delta\right), \ v \in \mathrm{B}\left(G(\bar{x}), \zeta/3\right).$$

This completes the proof. $\square$

Further examples of multifunctions satisfying Definition 2.2 can be found within the class of convex multifunctions, that plays a leading role in the present work. Let us recall that a set-valued mapping $G : \mathbb{X} \rightrightarrows \mathbb{Y}$ between Banach spaces is said to be convex if $\operatorname{gph} G$ is a convex set. Equivalently, G is convex iff

$$tG(x_1) + (1-t)G(x_2) \subseteq G(tx_1 + (1-t)x_2), \quad \forall t \in [0, 1], \quad \forall x_1, \, x_2 \in \mathbb{X},$$

with the convention that $\varnothing + S = \varnothing = t\varnothing$, for every $S \subseteq \mathbb{Y}$ and $t \in \mathbb{R}$ (see [2]).

Whenever a convex multifunction $G : \mathbb{X} \rightrightarrows \mathbb{Y}$ is also positively homogeneous, i.e.

$$\mathbf{0} \in G(\mathbf{0}) \qquad \text{and} \qquad G(\lambda x) = \lambda G(x), \quad \forall \lambda > 0, \forall x \in \mathbb{X},$$

it is called sublinear or, according to [2, 24, 23], a convex process. In other terms, sublinear set-valued mappings are characterized by having a cone in $\mathbb{X} \times \mathbb{Y}$ as their graph. During the 70-ies and the 80-ies, they have been the subject of deep investigations in convex and nonsmooth analysis. In particular, the study of their regularity properties has revealed that the value $\operatorname{reg} G(\mathbf{0}|\mathbf{0})$ plays a crucial role in understanding their special behaviour. More precisely, it is known that

$$\operatorname{reg} G(\mathbf{0}|\mathbf{0}) = \|G^{-1}\|^{-},$$

where

$$\|H\|^{-} = \sup_{x \in \mathbb{B}} \inf_{y \in H(x)} \|y\| = \sup_{x \in \mathbb{B}} \operatorname{dist}\left(\mathbf{0}, H(x)\right),$$

is the so-called inner norm of a positively homogeneous set-valued mapping $H : \mathbb{X} \rightrightarrows \mathbb{Y}$ (see [8, 9, 16]). Furthermore, it has been shown that for any sublinear mapping $G : \mathbb{X} \rightrightarrows \mathbb{Y}$ with closed graph it results in

$$\operatorname{reg} G(\bar{x}|\bar{y}) \leq \operatorname{reg} G(\mathbf{0}|\mathbf{0}), \quad \forall (\bar{x}, \bar{y}) \in \operatorname{gph} G \tag{4}$$

(see, for instance, [9]). In terms of the metric regularity notion introduced in Definition 2.2, such a property can be restated as follows.

Proposition 2.2 *Let $G : \mathbb{X} \rightrightarrows \mathbb{Y}$ be a sublinear set-valued mapping between real Banach spaces and let $(\bar{x}, \bar{y}) \in \operatorname{gph} G$. If $\operatorname{reg} G(\mathbf{0}|\mathbf{0}) < \infty$, then G is metrically regular at $\bar{x}$, for $G(\bar{x})$, and it holds*

$$\operatorname{reg} G(\bar{x}) \leq \operatorname{reg} G(\mathbf{0}|\mathbf{0}) = \operatorname{reg} G(\mathbf{0}).$$

After the works of Lyusternik, Graves, Robinson and Milyutin, it was well understood that the regularity property is stable under additive perturbations with locally Lipschitz mappings, provided that the Lipschitz modulus is small enough. The following result provides a quantitative description of such a persistence phenomenon (see [8, 9, 16]).

Theorem 2.1 (estimate for Lipschitz perturbations) *Consider a mapping* $G : \mathbb{X} \rightrightarrows \mathbb{Y}$ *and* $(\bar{x}, \bar{y}) \in \operatorname{gph} G$*, at which* $\operatorname{gph} G$ *is locally closed, and a mapping* $f : \mathbb{X} \longrightarrow \mathbb{Y}$*. If* $\operatorname{reg} G(\bar{x}|\bar{y}) < \kappa < \infty$ *and* $\operatorname{lip} f(\bar{x}) < \lambda < \kappa^{-1}$*, then*

$$\operatorname{reg}(f + G)(\bar{x}|f(\bar{x}) + \bar{y}) < \frac{1}{\kappa^{-1} - \lambda}.$$

In the next lemma, a uniform behaviour of the metric regularity property as given in Definition 2.2 in the presence of additive Lipschitz perturbations is obtained. It will be exploited in the proof of the main result.

Lemma 2.3 *Let* $G : \mathbb{X} \rightrightarrows \mathbb{Y}$ *be a set-valued mapping, with* $\operatorname{gph} G$ *locally closed, and let* $f : \mathbb{X} \longrightarrow \mathbb{Y}$ *and let* $\bar{x} \in \operatorname{dom} G$*. Suppose that* G *is metrically regular at* $\bar{x}$*, for* $G(\bar{x})$*,* f *is locally Lipschitz near* $\bar{x}$*, and*

$$\operatorname{reg} G(\bar{x}) < \operatorname{lip} f(\bar{x})^{-1}. \tag{5}$$

Then, the set-valued mapping $F = f + G$ *is metrically regular at* $\bar{x}$*, for* $F(\bar{x}) = f(\bar{x}) + G(\bar{x})$*. Moreover*

$$\operatorname{reg} F(\bar{x}) \leq \frac{1}{\operatorname{reg} G(\bar{x})^{-1} - \operatorname{lip} f(\bar{x})}, \quad \forall y \in\ G(\bar{x}).$$

Proof. Fix an arbitrary $y \in G(\bar{x})$. By inequality (2), taking an arbitrary κ, with $\kappa > \operatorname{reg} G(\bar{x}) \geq \operatorname{reg} G(\bar{x}|y)$, one obtains

$$\operatorname{dist}\left(x, G^{-1}(v)\right) \leq \kappa \operatorname{dist}(v, G(x)), \quad \forall x \in \mathrm{B}(\bar{x}, \delta), \forall v \in \mathrm{B}(y, \zeta).$$

By proceeding as in the proof of Theorem 2.1 (Theorem 3.3 in [8]), it is possible to find values of $\tilde{\kappa}$, $\tilde{\delta}$ and $\tilde{\zeta}$, depending only on κ, δ and ζ (but not on y!) [2], such that

$$\operatorname{dist}\left(x, F^{-1}(v)\right) \leq \tilde{\kappa} \operatorname{dist}(v, F(x)), \quad \forall x \in \mathrm{B}\left(\bar{x}, \tilde{\delta}\right), \forall v \in \mathrm{B}\left(f(\bar{x}) + y, \tilde{\zeta}\right), \tag{6}$$

with $\tilde{\kappa} = (\kappa - \lambda)^{-1}$, for any $\lambda \in (\operatorname{lip} f(\bar{x}), 1/\operatorname{reg} G(\bar{x}))$. The infimum over all values of $\tilde{\kappa}$ such that inequality (6) holds true can be shown consequently not to exceed $(\operatorname{reg} G(\bar{x}|y)^{-1} - \operatorname{lip} f(\bar{x}))^{-1}$, and hence the value $(\operatorname{reg} G(\bar{x})^{-1} - \operatorname{lip} f(\bar{x}))^{-1}$. Since if $v \in \mathrm{B}\left(F(\bar{x}), \tilde{\zeta}\right)$, then a $y \in G(\bar{x})$ must exist such that $v \in \mathrm{B}\left(f(\bar{x}) + y, \tilde{\zeta}\right)$, one gets the validity of inequality (2). According to the definition of $\operatorname{reg} F(\bar{x})$, this completes the proof. □

3 The main result

One is now in a position to establish the following sufficient condition for the convexity of the images of small balls through a convex multifunction G perturbed by a $\mathrm{C}^{1,1}$ mapping f, which is the main result of the paper.

[2] In the proof of Theorem 3.3, the new constants for which inequlity (6) holds are expressed in terms of κ, δ and ζ only.

Theorem 3.1 *Let $G : \mathbb{X} \rightrightarrows \mathbb{Y}$ be a set-valued mapping between real Banach spaces, let $f : \Omega \longrightarrow \mathbb{Y}$ be a mapping defined on an open set Ω and let x_0 and $r > 0$ such that $\mathrm{B}\,(x_0, r) \subseteq \Omega \cap \mathrm{dom}\, G$. Suppose that:*
(i) $(\mathbb{X}, \|\cdot\|)$ is of class UC_2, having second order modulus of convexity with some constant $c > 0$;
(ii) $f \in \mathrm{C}^{1,1}(\mathrm{int}\,\mathrm{B}\,(x_0, r))$;
(iii) G is a closed and convex multifunction;
(iv) G is upper semicontinuous (for short, u.s.c.) at x_0;
(v) G is metrically regular at x_0, for $G(x_0)$, with regularity modulus such that

$$\mathrm{reg}\, G(x_0) < \|\mathrm{D}f(x_0)\|_{\mathcal{L}}^{-1}; \tag{7}$$

(vi) there exists $\tau > 0$ such that $F(\mathrm{B}\,(x_0, t))$ is closed for every $t \in [0, \tau]$.
Then, there exists $\epsilon_0 > 0$, such that $F(\mathrm{B}\,(x_0, \epsilon))$ is convex, for every $\epsilon \in [0, \epsilon_0]$.

Proof. As already remarked, since $f \in \mathrm{C}^{1,1}(\mathrm{int}\,\mathrm{B}\,(x_0, r))$, then it is $\mathrm{lip}\, f(x_0) = \|\mathrm{D}f(x_0)\|_{\mathcal{L}} < \infty$. According to hypothesis (v), G is metrically regular at x_0, for $G(x_0)$, and condition (5) takes place. Thus, by virtue of Lemma 2.3, the set-valued mapping $F = f + G$ is metrically regular at x_0, for each $f(x_0) + y$, with $y \in G(x_0)$, that is there exist $\delta > 0$ and $\zeta > 0$ such that

$$\mathrm{dist}\left(x, F^{-1}(v)\right) \le \kappa\, \mathrm{dist}\,(v, F(x)), \quad \forall x \in \mathrm{B}\,(x_0, \delta), \ \forall v \in \mathrm{B}\,(f(x_0) + y, \zeta), \tag{8}$$

for any $\kappa > (\mathrm{reg}\, G(x_0)^{-1} - \|\mathrm{D}f(x_0)\|_{\mathcal{L}})^{-1}$, and it holds

$$\mathrm{reg}\, F(x_0 | f(x_0) + y) \le \frac{1}{\mathrm{reg}\, G(x_0)^{-1} - \|\mathrm{D}f(x_0)\|_{\mathcal{L}}}.$$

Recall that the constants appearing in inequality (8) remain the same for every $y \in G(x_0)$. Then, corresponding to $\zeta/4$, as a consequence of hypothesis (ii), by continuity of f at x_0, there is $\delta_1 > 0$ such that

$$f(x) \in \mathrm{B}\,(f(x_0), \zeta/4), \quad \forall x \in \mathrm{B}\,(x_0, \delta_1).$$

Again, by upper semicontinuity of G at x_0 (hypothesis (iv)), corresponding to $\zeta/4$, there is $\delta_2 > 0$ such that

$$G(x) \subseteq \mathrm{B}\,(G(x_0), \zeta/4), \quad \forall x \in \mathrm{B}\,(x_0, \delta_2).$$

Consequently, take ϵ_0 in such a way that

$$0 < \epsilon_0 < \left\{ \delta,\, \delta_1,\, \delta_2,\, \tau,\, r,\, \frac{4c(\mathrm{reg}\, G(x_0)^{-1} - \|\mathrm{D}f(x_0)\|_{\mathcal{L}})}{\mathrm{Lip}(\mathrm{D}f, \mathrm{int}\,\mathrm{B}\,(x_0, r)) + 1} \right\}. \tag{9}$$

In the case $\epsilon = 0$ the thesis becomes trivial, because $F(x_0) = f(x_0) + G(x_0)$ is convex as a sum of the convex sets $\{f(x_0)\}$ and $G(x_0)$ (the latter is convex as a consequence of hypothesis (iii)). Now, fix $\epsilon \in (0, \epsilon_0]$. Since it is $\epsilon < \tau$ and hence, according to hypothesis (vi) set $F(\mathrm{B}\,(x_0, \epsilon))$ is closed, to show that this set is convex it suffices to prove that, whenever $y_1, y_2 \in F(\mathrm{B}\,(x_0, \epsilon))$, it happens also that

$$\bar{y} = \frac{y_1 + y_2}{2} \in F(\mathrm{B}\,(x_0, \epsilon)).$$

The fact that $y_1 \in F(\mathrm{B}\,(x_0, \epsilon))$ implies the existence of $x_1 \in \mathrm{B}\,(x_0, \epsilon)$ such that $y_1 \in F(x_1) = f(x_1) + G(x_1)$, and hence the existence of $v_1 \in G(x_1)$ such that $y_1 = f(x_1) + v_1$. Analogously, the fact that $y_2 \in F(\mathrm{B}\,(x_0, \epsilon))$ implies the existence of $x_2 \in \mathrm{B}\,(x_0, \epsilon)$ and $v_2 \in G(x_2)$, such that $y_2 = f(x_2) + v_2$. Set

$$\bar{v} = \frac{v_1 + v_2}{2} \qquad \text{and} \qquad \bar{x} = \frac{x_1 + x_2}{2}.$$

If $\bar{y} \in F(\bar{x}) \subseteq F(\mathrm{B}(x_0, \epsilon))$ the argument is finished. Otherewise, it is $\mathrm{dist}(\bar{y}, F(\bar{x})) > 0$ because $F(\bar{x})$ is closed. Notice that, since it is $\epsilon < \delta_1$, one has $f(x_1),\ f(x_2) \in \mathrm{B}(f(x_0), \zeta/4)$ and hence

$$\frac{f(x_1) + f(x_2)}{2} \in \mathrm{B}(f(x_0), \zeta/4) \,. \tag{10}$$

Since it is $\epsilon < \delta_2$, one has that $v_1,\ v_2 \in \mathrm{B}(G(x_0), \zeta/4)$. The fact that G is a convex multifunction implies that $G(x_0)$ is convex, and so is function $v \mapsto \mathrm{dist}(v, G(x_0))$, with the consequence that

$$\bar{v} \in \mathrm{B}(G(x_0), \zeta/4) \,.$$

This means that there exists $y_0 \in G(x_0)$ such that $d(\bar{v}, y_0) < \zeta/2$. Then, from inequality (10) it follows

$$\begin{aligned} d(\bar{y}, f(x_0) + y_0) \quad &= \quad \left\| \frac{f(x_1) + f(x_2)}{2} + \bar{v} - (f(x_0) + y_0) \right\| \leq \left\| \frac{f(x_1) + f(x_2)}{2} - f(x_0) \right\| + \|\bar{v} - y_0\| \\ &\leq \quad \frac{\zeta}{4} + \frac{\zeta}{2} < \zeta. \end{aligned}$$

The above inequalities show that $\bar{x} \in \mathrm{B}(x_0, \delta)$ and $\bar{y} \in \mathrm{B}(f(x_0) + y_0, \zeta)$, so inequality (8) applies, namely for any $\kappa > (\mathrm{reg}\, G(x_0)^{-1} - \|\mathrm{D}f(x_0)\|_{\mathcal{L}})^{-1}$ it holds

$$\mathrm{dist}\left(\bar{x}, F^{-1}(\bar{y})\right) \leq \kappa \,\mathrm{dist}(\bar{y}, F(\bar{x})) \,.$$

As a consequence of the last inequality, there exists $\hat{x} \in F^{-1}(\bar{y})$ such that

$$d(\bar{x}, \hat{x}) < \frac{2\,\mathrm{dist}(\bar{y}, F(\bar{x}))}{\mathrm{reg}\, G(x_0)^{-1} - \|\mathrm{D}f(x_0)\|_{\mathcal{L}}}. \tag{11}$$

Now, observe that, by an obvious translation of vectors, one obtains

$$\mathrm{dist}(\bar{y}, F(\bar{x})) = \mathrm{dist}\left(\frac{f(x_1) + f(x_2)}{2} + \bar{v}, f(\bar{x}) + G(\bar{x})\right) = \mathrm{dist}\left(\frac{f(x_1) + f(x_2)}{2} - f(\bar{x}), G(\bar{x}) - \bar{v}\right).$$

Since, by convexity of $\mathrm{gph}\, G$, it is

$$\bar{v} \in \frac{G(x_1) + G(x_2)}{2} \subseteq G(\bar{x}),$$

in the light of Lemma 2.2 it results in

$$\begin{aligned} \mathrm{dist}(\bar{y}, F(\bar{x})) \quad &\leq \quad \mathrm{dist}\left(\frac{f(x_1) + f(x_2)}{2} - f(\bar{x}), \frac{G(x_1) + G(x_2)}{2} - \bar{v}\right) \leq \left\| \frac{f(x_1) + f(x_2)}{2} - f(\bar{x}) \right\| \\ &\leq \quad \frac{\mathrm{Lip}(\mathrm{D}f, \mathrm{int}\, \mathrm{B}(x_0, r))}{8} \|x_1 - x_2\|^2. \end{aligned}$$

From inequality (11), recalling that

$$\epsilon < \frac{4c(\mathrm{reg}\, G(x_0)^{-1} - \|\mathrm{D}f(x_0)\|_{\mathcal{L}})}{\mathrm{Lip}(\mathrm{D}f, \mathrm{int}\, \mathrm{B}(x_0, r)) + 1},$$

one obtains

$$d(\hat{x}, \bar{x}) < \frac{\mathrm{Lip}(\mathrm{D}f, \mathrm{B}(x_0, r))}{4(\mathrm{reg}\, G(x_0)^{-1} - \|\mathrm{D}f(x_0)\|_{\mathcal{L}})} \|x_1 - x_2\|^2 < \frac{c}{\epsilon} \|x_1 - x_2\|^2,$$

whence it follows that

$$\hat{x} \in \mathrm{B}\left(\bar{x}, \frac{c}{\epsilon} \|x_1 - x_2\|^2\right).$$

By the uniform convexity of $\mathbb{X}$, with modulus of second order of constant c, the last inclusion is known to imply that $\hat{x} \in \mathrm{B}(x_0, \epsilon)$, according to Lemma 2.1. Thus

$$\bar{y} \in f(\hat{x}) + G(\hat{x}) \subseteq F(\mathrm{B}(x_0, \epsilon)).$$

The arbitrariness of $\epsilon \in (0, \epsilon_0]$ completes the proof. □

Remark 3.1 (r_1) The reader should notice that Theorem 3.1 has a local nature. Therefore hypothesis (iii), the only global one, can be actually weakened by assuming G to be locally closed convex near x_0 and $G(x_0)$, i.e. that there exists $r > 0$ such that $\mathrm{gph}\, G \cap [\mathrm{B}(x_0, r) \times \mathrm{B}(G(x_0), r)]$ is closed and convex. A perusal of the arguments in the proof confirms the validity of such a refinement.

(r_2) Whenever $G : \mathbb{X} \rightrightarrows \mathbb{Y}$ is, in particular, a closed sublinear set-valued mapping, then in the light of the global metric regularity property recalled in Proposition 2.2, hypothesis (v) takes the simpler form: G is metrically regular at $\mathbf{0}$, for $\mathbf{0}$, and $\mathrm{reg}\, G(\mathbf{0}|\mathbf{0}) < \|\mathrm{D}f(x_0)\|_{\mathcal{L}}^{-1}$.

(r_3) As a consequence of the metric regularity of $F = f + G$ at x_0, for $F(x_0)$, it follows that if $x \in \mathrm{int}\, \mathrm{B}(x_0, \epsilon)$, with $\epsilon \in (0, \epsilon_0]$, and $y \in F(x)$, then $y \in \mathrm{int}\, F(\mathrm{B}(x_0, \epsilon))$. Thus, if denoting by $\mathrm{bd}\, A$ the boundary of a subset A, whenever $y \in \mathrm{bd}\, F(\mathrm{B}(x_0, \epsilon))$ and $x \in F^{-1}(y)$, one obtains that $x \notin \mathrm{int}\, \mathrm{B}(x_0, \epsilon)$, namely $x \in \mathrm{bd}\, \mathrm{B}(x_0, \epsilon)$. In particular, one has that $\mathrm{int}\, F(\mathrm{B}(x_0, \epsilon)) \neq \varnothing$.

(r_4) The strict inequality appearing in (7) is essential and can not be relaxed by a non strict one, even in very simple cases, as illustrated by the counterexample below.

Example 3.1 Let $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ and $G : \mathbb{R} \rightrightarrows \mathbb{R}^2$ be given by

$$f(x) = (0,\, x^2) \qquad \text{and} \qquad G(x) = \{(x, x)\},$$

respectively, and let $x_0 = 0$, with $\mathbb{R}$ and $\mathbb{R}^2$ equipped with their usual Euclidean structure. Then, it results in

$$F(x) = f(x) + G(x) = \{(x, x^2 + x)\}.$$

Notice that $f \in \mathrm{C}^{1,1}(\mathbb{R})$ and G is a convex process. Throught elementary calculations, one finds $\|\mathrm{D}f(0)\| = 0$ and, since G is not onto, $\mathrm{reg}\, G(0) = +\infty$. In other words, the stric inequality (7) is not true, being replaced by an equality. As one easily checks, all remaining hypotheses of Theorem 3.1 are fulfilled. In this case the thesis fails to be true. Indeed, the image of a ball $\mathrm{B}(0, \epsilon) = [-\epsilon, \epsilon]$ through F is the set

$$F([-\epsilon, \epsilon]) = \{(x, x^2 + x) \in \mathbb{R}^2 :\ -\epsilon \leq x \leq \epsilon\},$$

that fails to be convex, for every $\epsilon > 0$.

From Theorem 3.1 one can derive, as a special case, a sufficient condition for the convexity of images of small balls, around a regular point, which is known as a Polyak's convexity principle.

Corollary 3.1 (Polyak convexity principle) *Let $f : \mathbb{X} \longrightarrow \mathbb{Y}$ be a mapping between real Banach spaces, let Ω be an open subset of $\mathbb{X}$, let $x_0 \in \Omega$, and $r > 0$ such that $\mathrm{B}(x_0, r) \subseteq \Omega$. Suppose that:*
(i) $(\mathbb{X}, \|\cdot\|)$ is of class UC_2;
(ii) $f \in \mathrm{C}^{1,1}(\Omega)$ and $\mathrm{D}f(x_0) \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$ is onto.
Then, there exists $\epsilon_0 \in (0, r)$ such that $f(\mathrm{B}(x_0, \epsilon))$ is convex, for every $\epsilon \in [0, \epsilon_0]$.

Proof. Observe that, under the current hypotheses, the mapping $x \mapsto \{\mathrm{D}f(x_0)[x]\}$ is a closed sublinear multifunction, which is u.s.c. at x_0, as $\mathrm{D}f(x_0) \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$. According to the Banach-Schauder theorem, the fact that $\mathrm{D}f(x_0)$ is onto is equivalent to its global metric regularity, and it holds

$$\operatorname{reg} \mathrm{D}f(x_0)(\mathbf{0}|\mathbf{0}) = \|\mathrm{D}f(x_0)^{-1}\|^{-} < \infty$$

(here $\mathrm{D}f(x_0)^{-1}$ denotes the multivalued inverse of $\mathrm{D}f(x_0)$). Therefore, it remains to set

$$h = f - \mathrm{D}f(x_0),$$

so that $f = h + \mathrm{D}f(x_0)$ can be expressed as a perturbation of $\mathrm{D}f(x_0)$. Clearly $h \in \mathrm{C}^{1,1}(\operatorname{int} \mathrm{B}(x_0, r))$ and $\mathrm{D}h(x_0) = \mathbf{0} \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$, so, according to the convention made, condition (7) is fulfilled, independently of the value of $\|\mathrm{D}f(x_0)^{-1}\|^{-}$. Finally, in Lemma 2.10 of [25] the closedness of $f(\mathrm{B}(x_0, t))$, for every $t \in [0, \tau]$, has been shown to come as a consequence of the metric regularity of f at x_0, which is in turn a consequence of the surjectivity of $\mathrm{D}f(x_0)$, as it is know by the Lyusternik-Graves theorem. Thus, Theorem 3.1 applies. □

4 An application to set-valued optimization

In this section an application of the main result is presented, which concerns set-valued optimization. This is a rather recent branch of optimization, focusing on problems whose objective (or cost) function are set-valued mappings. Some motivating examples, coming from applications to mathematical economics as well as from theoretical issues in vector optimization, fuzzy programming and robust optimization, are described, for instance, in [3, 4, 12].

In what follows, let us assume that a vector objective function $q : \mathbb{X} \longrightarrow \mathbb{Y}$, acting in abstract spaces, is given as a problem datum, along with a partial ordering $\leq_C$ on its range space, which is defined by a proper, convex, pointed and closed cone $C \subset \mathbb{Y}$. In real-world scenarios, it may happen that the value of q is affected by noise effects, due to approximations, errors and/or incompleteness in measurement and informations. As a result, instead of a unique vector cost $q(x)$ corresponding to a chosen strategy x in the decision space $\mathbb{X}$, one has to deal with a set of several vectors in $\mathbb{Y}$. This situation can be formalized by assuming that q is perturbed by adding a given set-valued mapping $Q : \mathbb{X} \rightrightarrows \mathbb{Y}$, leading to a set-valued objective $\Phi = q + Q$. The resulting (unconstrained) optimization problem is

$$(\mathcal{SP}) \qquad \text{minimize}_C \ \Phi(x) \quad \text{over } \Omega,$$

where Ω is a nonempty open subset of $\mathbb{X}$. Throughout the present section, it will be assumed that $\operatorname{dom} \Phi \supseteq \Omega$.

For such a problem several solution concepts have been proposed. Following a vector based approach, according to [12] a pair $(\bar{x}, \bar{y}) \in \operatorname{gph} \Phi$ is said to be a C-efficient pair for problem $(\mathcal{SP})$ if

$$(\bar{y} - C) \cap \Phi(\Omega) = \{\bar{y}\}.$$

Notice that $\bar{y}$ is a C-minimal element of $\Phi(\Omega)$ with respect to the partial order relation $\leq_C$. In this context, as a consequence of Theorem 3.1, the existence of C-efficient pairs of localizations of problem $(\mathcal{SP})$ is established. Given a point $x_0 \in \Omega$ and $\epsilon > 0$, by a localization of problem $(\mathcal{SP})$ the following constrained set-valued minimization problem is meant

$$(\mathcal{SP}_{x_0,\epsilon}) \qquad \text{minimize}_C \ \Phi(x) \quad \text{subject to } x \in \mathrm{B}(x_0, \epsilon).$$

The following technical lemma will be employed in the proof of the next result.

Lemma 4.1 *Let $q : \mathbb{X} \longrightarrow \mathbb{Y}$ and $Q : \mathbb{X} \rightrightarrows \mathbb{Y}$ be given. Suppose that q is continuous at $x_0 \in \mathbb{X}$, Q is u.s.c. at x_0 and set $Q(x_0)$ is bounded. The the set-valued mapping $\Phi = q + Q$ is locally bounded around x_0, i.e. there exist a bounded set $W \subset \mathbb{Y}$ and $r > 0$ such that*

$$\Phi(x) \subseteq W, \quad \forall x \in \mathrm{B}\,(x_0, r)\,.$$

Proof. By the continuity of q at x_0, corresponding to $\eta > 0$ there exists $r_q > 0$ such that

$$q(x) \in \mathrm{B}\,(q(x_0), \eta)\,, \quad \forall x \in \mathrm{B}\,(x_0, r_q)\,.$$

By the upper semicontinuity of Q at x_0, corresponding to $\eta > 0$ there exists $r_Q > 0$ such that

$$Q(x) \subseteq \mathrm{int}\,\mathrm{B}\,(Q(x_0), \eta)\,, \quad \forall x \in \mathrm{B}\,(x_0, r_Q)\,.$$

Notice that, since $Q(x_0)$ is bounded, also $\mathrm{B}\,(Q(x_0), \eta)$ is bounded. Thus, taking $r_\Phi = \min\{r_q, r_Q\}$, it holds

$$\Phi(x) = q(x) + Q(x) \subseteq \mathrm{B}\,(q(x_0), \eta) + \mathrm{B}\,(Q(x_0), \eta)\,, \quad \forall x \in \mathrm{B}\,(x_0, r_\Phi)\,,$$

and hence it suffices to set $W = \mathrm{B}\,(q(x_0), \eta) + \mathrm{B}\,(Q(x_0), \eta)$. □

Proposition 4.1 *Suppose that the data of problem $(\mathcal{SP})$ satisfy the following assumptions:*
(a_1) $(\mathbb{X}, \|\cdot\|)$ is of class UC_2 and $(\mathbb{Y}, \|\cdot\|)$ is reflexive;
(a_2) $q \in \mathrm{C}^{1,1}(\mathrm{int}\,\mathrm{B}\,(x_0, r))$, for some $x_0 \in \mathbb{X}$ and $r > 0$, such that $\mathrm{B}\,(x_0, r) \subseteq \Omega$;
(a_3) Q is locally closed and convex multifunction near x_0 and $Q(x_0)$;
(a_4) set $Q(x_0)$ is bounded and the set-valued mapping Q is u.s.c. at x_0;
(a_5) Q is metrically regular at x_0, for $Q(x_0)$, with regularity modulus such that

$$\mathrm{reg}\, Q(x_0) < \|\mathrm{D}q(x_0)\|_{\mathcal{L}}^{-1};$$

(a_6) there exists $\tau > 0$ such that $\Phi(\mathrm{B}\,(x_0, t))$ is closed for every $t \in [0, \tau]$.
Then, there exists $\epsilon_0 > 0$, such that for every $\epsilon \in (0, \epsilon_0]$ problem $(\mathcal{SP}_{x_0,\epsilon})$ admits a C-efficient pair $(x_\epsilon, y_\epsilon) \in \mathrm{bd}\,\mathrm{B}\,(x_0, \epsilon) \times \Phi(x_\epsilon)$.

Proof. Under the above hypotheses it is possible to apply Theorem 3.1. According to it, there exists a positive ϵ_0 such that for every $\epsilon \in (0, \epsilon_0]$ the image $\Phi(\mathrm{B}\,(x_0, \epsilon))$ is a convex subset of $\mathbb{Y}$. Now, fix any $\epsilon \in (0, \epsilon_0]$ and consider the corresponding localized problem $(\mathcal{SP}_{x_0,\epsilon})$. Observe that $\Phi(\mathrm{B}\,(x_0, \epsilon))$ is compact with respect to the weak topology in $\mathbb{Y}$. Indeed, as it is norm closed and convex, it is also weakly closed. Besides, since $Q(x_0)$ is bounded, by virtue of Lemma 4.1 the mapping Φ turns out to be locally bounded around x_0. Thus, up to a reduction in the value of ϵ_0, one can assume that $\Phi(\mathrm{B}\,(x_0, \epsilon))$ is bounded. So the reflexivity of $(\mathbb{Y}, \|\cdot\|)$ entails that $\Phi(\mathrm{B}\,(x_0, \epsilon))$ is weakly compact. By virtue of Theorem 6.5 (a) in [11], there exists an element $y_\epsilon \in \Phi(\mathrm{B}\,(x_0, \epsilon))$, which is C-minimal. This means that there is $x_\epsilon \in \mathrm{B}\,(x_0, \epsilon)$, with $y_\epsilon \in \Phi(x_\epsilon)$, such that (x_ϵ, y_ϵ) is a C-efficient pair for $(\mathcal{SP}_{x_0,\epsilon})$. Observe that, as a C-minimal element of $\Phi(\mathrm{B}\,(x_0, \epsilon))$, y_ϵ must be in $\mathrm{bd}\,\Phi(\mathrm{B}\,(x_0, \epsilon))$. As noted in Remark 3.1 (r_3), since $x_\epsilon \in \Phi^{-1}(y_\epsilon)$, it is $x_\epsilon \in \mathrm{bd}\,\mathrm{B}\,(x_0, \epsilon)$. This completes the proof. □

Remark 4.1 As a comment to Proposition 4.1, it should be noted that its thesis is trivial if $\mathbb{X}$ and $\mathbb{Y}$ are finite-dimensional Euclidean spaces, because $\Phi(\mathrm{B}(x_0,\epsilon))$ is compact. In an abstract space setting, under the hypotheses of Proposition 4.1, it is possible to state that $\mathrm{B}(x_0,\epsilon)$ is weakly compact (recall Remark 2.1(r_3)). Nevertheless, since q may not be continuous with respect to the weak topologies, already the set $q(\mathrm{B}(x_0,\epsilon))$ may happen to be not weakly compact, in the absence of convexity assumptions.

The next result, which comes as a further consequence of Theorem 3.1, is an optimality condition useful for detecting solution pairs of $(\mathcal{SP}_{x_0,\epsilon})$. It can be regarded as a scalarization method, relying on the use of the following Lagrangian function $L:\mathbb{X}\times\mathbb{Y}^*\longrightarrow\mathbb{R}\cup\{-\infty\}$

$$L(x,y^*)=\langle y^*,q(x)\rangle+\inf_{y\in Q(x)}\langle y^*,y\rangle,$$

where $\mathbb{Y}^*$ denotes the dual space of $\mathbb{Y}$, whose null vector is marked by $\mathbf{0}^*$, and $\langle\cdot,\cdot\rangle:\mathbb{Y}^*\times\mathbb{Y}\longrightarrow\mathbb{R}$ denotes the canonical duality pairing $\mathbb{Y}^*$ with $\mathbb{Y}$. To formulate such a result, one needs to consider elements in the cone

$$C^+=\{y^*\in\mathbb{Y}^*:\ \langle y^*,y\rangle\geq 0,\quad \forall y\in C\}.$$

Proposition 4.2 *Under the hypotheses of Proposition 4.1, corresponding with any $\epsilon\in(0,\epsilon_0]$ and with a C-efficient pair (x_ϵ,y_ϵ), there exists $y^*_\epsilon\in C^+\backslash\{\mathbf{0}^*\}$ such that x_ϵ solves the scalar problem*

$$\text{minimize } L(x,y^*_\epsilon)\qquad \textit{subject to } x\in\mathrm{B}(x_0,\epsilon)\,.$$

Proof. By Proposition 4.1, there exists a pair $(x_\epsilon,y_\epsilon)\in\mathrm{bd}\,\mathrm{B}(x_0,\epsilon)\times\Phi(x_\epsilon)$, which is C-efficient for problem $(\mathcal{SP}_{x_0,\epsilon})$. Since $y_\epsilon\in\Phi(x_\epsilon)=q(x_\epsilon)+Q(x_\epsilon)$, there exists $v_\epsilon\in Q(x_\epsilon)$ such that $y_\epsilon=q(x_\epsilon)+v_\epsilon$. Recall that the set $\Phi(\mathrm{B}(x_0,\epsilon))$ is closed, convex and with nonempty interior (remember Remark 3.1 (r_3)). Since $y_\epsilon-C$ is convex and $(y_\epsilon-C)\cap\Phi(\mathrm{B}(x_0,\epsilon))=\{y_\epsilon\}$, the Heidelheit theorem applies. Consequently, there exist $y^*_\epsilon\in\mathbb{Y}^*\backslash\{\mathbf{0}^*\}$ and $\alpha\in\mathbb{R}$ such that

$$\langle y^*_\epsilon,y\rangle\leq\alpha,\quad \forall y\in y_\epsilon-C \tag{12}$$

and

$$\langle y^*_\epsilon,y\rangle\geq\alpha,\quad \forall y\in\Phi(\mathrm{B}(x_0,\epsilon)). \tag{13}$$

From inequality (12) it follows

$$\langle y^*_\epsilon,y_\epsilon\rangle-\langle y^*_\epsilon,y\rangle\leq\alpha,\quad \forall y\in C. \tag{14}$$

In particular, as it is $\mathbf{0}\in C$, one has $\langle y^*_\epsilon,y_\epsilon\rangle\leq\alpha$. On the other hand, as $y_\epsilon\in\Phi(\mathrm{B}(x_0,\epsilon))$, then from inequality (13), it is also $\langle y^*_\epsilon,y_\epsilon\rangle\geq\alpha$, whence it results in

$$\langle y^*_\epsilon,y_\epsilon\rangle=\alpha.$$

On account of the last equality, one sees that inequality (14) implies $y^*_\epsilon\in C^+$.

Now, recalling that $\Phi(x)=q(x)+Q(x)$, from inequality (13) one obtains for every $x\in\mathrm{B}(x_0,\epsilon)$

$$\langle y^*_\epsilon,q(x)\rangle+\langle y^*_\epsilon,y\rangle\geq\langle y^*_\epsilon,q(x_\epsilon)+v_\epsilon\rangle,\quad \forall y\in Q(x).$$

In particular, for $x = x_\epsilon$ it holds

$$\langle y_\epsilon^*, q(x_\epsilon)\rangle + \langle y_\epsilon^*, y\rangle \geq \langle y_\epsilon^*, q(x_\epsilon) + v_\epsilon\rangle, \quad \forall y \in Q(x_\epsilon).$$

This allows one to deduce that

$$\langle y_\epsilon^*, v_\epsilon\rangle = \min_{y\in Q(x_\epsilon)} \langle y_\epsilon^*, y\rangle.$$

According to the definition of L, one finds

$$L(x, y_\epsilon^*) = \langle y_\epsilon^*, q(x)\rangle + \inf_{y\in Q(x)} \langle y_\epsilon^*, y\rangle \geq \langle y_\epsilon^*, q(x_\epsilon)\rangle + \min_{y\in Q(x_\epsilon)} \langle y_\epsilon^*, y\rangle = L(x_\epsilon, y_\epsilon^*), \quad \forall x \in \mathrm{B}\,(x_0, \epsilon)\,.$$

This completes the proof. □

Remark 4.2 (r_1) It is worth noting that, under the assumptions of Proposition 4.2 the Lagrangian function L can be written

$$L(x, y^*) = \langle y^*, q(x)\rangle + \min_{y\in Q(x)} \langle y^*, y\rangle$$

in a neighbourhood of x_0. Indeed, recall that each element $y^* \in \mathbb{Y}^*$ is also weakly continuous. As already seen, since $Q(x_0)$ is bounded and Q is u.s.c. at x_0, Q turns out to be locally bounded. Therefore, in the reflexive space $(\mathbb{Y}, \|\cdot\|)$ each set $Q(x)$ is weakly compact, for x near x_0, with the consequence that y^* attains its minimum on it.

(r_2) A feature of Proposition 4.2 to be commented is that it establishes a scalarization condition which is typical in problems with convex graph objective, even though the graph of Φ is not necessarily convex. Indeed, according to Proposition 4.2, a C-efficient pair turns out to be a solution for a scalar problem involving the Lagrangian function L, what is more than a mere stationarity condition for L. This happens by virtue of the convexity principle, which enables one to exploit the "hidden convexity" of the problem. In this concern, notice that, even if the function $x \mapsto \inf_{y\in Q(x)} \langle y^*, y\rangle$ is convex under the hypotheses of Proposition 4.2, the function $x \mapsto L(x, y^*)$ may lose this property, owing to the additional term $\langle y^*, q(x)\rangle$.

References

[1] Aliprantis, C.D.; Border, K.C. *Infinite dimensional analysis. A hitchhiker's guide.* Springer, Berlin, 2006.

[2] Aubin, J.-P.; Frankowska, H. *Set-valued analysis.* Systems & Control: Foundations & Applications, 2. Birkhäuser Boston, Inc., Boston, MA, 1990.

[3] Bao, T.Q.; Mordukhovich, B.S. *Set-valued optimization in welfare economics.* Advances in mathematical economics. Volume 13, 113–153, Adv. Math. Econ., 13, Springer, Tokyo, 2010.

[4] Bao, T.Q.; Mordukhovich, B.S. *Refined necessary conditions in multiobjective optimization with applications to microeconomic modeling.* Discrete Contin. Dyn. Syst. **31** (2011), no. 4, 1069–1096.

[5] Bobylev, N.A.; Emelyanov, S.V.; Korovin, S.K. *Convexity of images of convex sets under smooth maps.* Comput. Math. Model. **15** (2004), no. 3, 213–222.

[6] Borwein, J.M.; Zhu, Q.J. *Techniques of variational analysis.* CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 20. Springer-Verlag, New York, 2005.

[7] Clarkson, J.A. *Uniformly convex spaces.* Trans. Amer. Math. Soc. **40** (1936), no. 3, 396–414.

[8] Dontchev, A.L.; Lewis, A.S.; Rockafellar, R. T. *The radius of metric regularity.* Trans. Amer. Math. Soc. **355** (2003), no. 2, 493–517.

[9] Dontchev, A.L.; Rockafellar, R. T. *Implicit functions and solution mappings. A view from variational analysis.* Springer Monographs in Mathematics. Springer, Dordrecht, 2009.

[10] Fabian, M.; Habala, P.; Hájek, P.; Montesinos Santalucía, V.; Pelant, J.; Zizler, V. *Functional analysis and infinite-dimensional geometry.* Springer-Verlag, New York, 2001.

[11] Jahn, J. *Vector optimization. Theory, applications, and extensions.* Springer-Verlag, Berlin, 2004.

[12] Khan, A.A.; Tammer, K.; Zălinescu, C. *Set-valued optimization. An introduction with applications.* Vector Optimization. Springer, Heidelberg, 2015.

[13] Klatte, D.; Kummer, B. *Nonsmooth equations in optimization. Regularity, calculus, methods and applications.* Kluwer Academic Publishers, Dordrecht, 2002.

[14] Lyapunov, A.A. *Sur les fonctions-vecteurs complètement additives.* Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **4** (1940), 465–478.

[15] Megginson, R.E. *An Introduction to Banach Space Theory.* Springer-Verlag, New York, 1998.

[16] Mordukhovich, B.S. *Variational Analysis and Generalized Differentiation I: Basic Theory.* Springer-Verlag, Berlin Heidelberg, 2006.

[17] Olech, C. *The Lyapunov theorem: its extensions and applications.* Methods of nonconvex analysis (Varenna, 1989), 84–103, Lecture Notes in Math., 1446, Springer, Berlin, 1990.

[18] Polyak, B.T. *Convexity of nonlinear image of a small ball with applications to optimization.* Wellposedness in optimization and related topics (Gargnano, 1999). Set-Valued Anal. **9** (2001), no. 1-2, 159–168.

[19] Polyak, B.T. *The convexity principle and its applications.* Bull. Braz. Math. Soc. (N.S.) **34** (2003), no. 1, 59–75.

[20] Rockafellar, R.T.; Wets, R.J.-B. *Variational Analysis*, Springer, Berlin Heidelberg, 1998.

[21] Reißig, G. *Convexity of the reachable sets of control systems.* Autom. Remote Control **68** (2007), no. 9, 1527–1543.

[22] Reißig, G. *Computing abstractions of nonlinear systems.* IEEE Trans. Automat. Control **56** (2011), no. 11, 2583–2598.

[23] Robinson, S.M. *Normed convex processes.* Trans. Amer. Math. Soc. **174** (1972), 127–140.

[24] Rockafellar, R.T. *Convex analysis.* Princeton Mathematical Series, No. 28 Princeton University Press, Princeton, N.J. 1970.

[25] Uderzo, A. *On the Polyak convexity principle and its application to variational analysis.* Nonlinear Anal. **91** (2013), 60–71.